4
Applications

4.1 Ordinary Differential Equations

There is a strong connection between the implicit function theorem and the theory of differential equations. This is true even from the historical point of view, for Picard's iterative proof of the existence theorem for ordinary differential equations inspired Goursat to give an iterative proof of the implicit function theorem (see Goursat [Go 03]). In the mid-twentieth century, John Nash pioneered the use of a sophisticated form of the implicit function theorem in the study of partial differential equations. We will discuss Nash's work in Section 6.4. In this section, we limit our attention to ordinary (rather than partial) differential equations because the technical details are then so much simpler. Our plan is first to show how a theorem on the existence of solutions to ordinary differential equations can be used to prove the implicit function theorem. Then we will go the other way by using a form of the implicit function theorem to prove an existence theorem for differential equations.

A typical existence theorem for ordinary differential equations is the following fundamental result[1] (see for example, Hurewicz [Hu 64]):

Theorem 4.1.1 (Picard) *If $F(t, x)$, $(t, x) \in \mathbf{R}\times\mathbf{R}^N$, is continuous in the $(N+1)$-dimensional region $(t_0 - a, t_0 + a) \times \mathbf{B}(x_0, r)$, then there exists a solution $x(t)$*

[1]This fundamental theorem is commonly known as *Picard's existence and uniqueness theorem*. The classical proof uses a method that has come to be known as the *Picard iteration technique*. See [Pi 93].

of

$$\frac{dx}{dt} = F(t, x), \quad x(t_0) = x_0, \tag{4.1}$$

defined over an interval $(t_0 - h, t_0 + h)$.

Remark 4.1.2 The solution of (4.1) need not be unique if F is only continuous. For example, the problem of finding $x(t)$ satisfying $x' = x^{2/3}$, $x(0) = 0$, has the two solutions $x \equiv 0$ and $x(t) = (t/3)^3$ To guarantee that the solution of (4.1) is unique, it is sufficient to assume additionally that F satisfies a Lipschitz condition as a function of x

We can give an alternative proof of the implicit function theorem as a corollary of Theorem 4.1.1.

Theorem 4.1.3 *Suppose that* $U \subset \mathbf{R}^{N+1}$ *and that* $H : U \to \mathbf{R}$ *is* C^1 *If* $H(t_0, x_0) = 0$, $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^N$, *and the* $N \times N$ *matrix*

$$\left(\frac{\partial H_i}{\partial x_j}(t_0, x_0)\right)_{i,j=1,2,\ldots,N}$$

is nonsingular, then there exists an open interval $(t_0 - h, t_0 + h)$ *and a continuously differentiable function* $\phi : (t_0 - h, t_0 + h) \to \mathbf{R}^N$ *such that* $\phi(t_0) = x_0$ *and*

$$H(t, \phi(t)) = 0.$$

Proof. We consider the case $N = 1$ in some detail. First, choose $a, r > 0$ so that $(t_0 - a, t_0 + a) \times (x_0 - r, x_0 + r) \subseteq U$ and $(\partial H/\partial x)(t, x)$ is nonvanishing on $(t_0 - a, t_0 + a) \times (x_0 - r, x_0 + r)$. Then define $F : (t_0 - a, t_0 + a) \times (x_0 - r, x_0 + r) \to \mathbf{R}$ by setting

$$F(t, x) = -\frac{\partial H}{\partial t}(t, x) \Big/ \frac{\partial H}{\partial x}(t, x). \tag{4.2}$$

Since F is continuous, we can apply Theorem 4.1.1 to conclude that there exists a solution of the problem

$$\frac{dx}{dt} = F(t, x), \quad x(t_0) = x_0$$

defined on an interval $(t_0 - h, t_0 + h)$. We define $\phi : (t_0 - h, t_0 + h) \to \mathbf{R}$ by setting

$$\phi(t) = x(t).$$

Note that

$$\phi(t_0) = x_0 \tag{4.3}$$

and

$$\begin{aligned}\phi'(t) &= \frac{dx}{dt}(t) = F(t, x(t)) \\ &= -\frac{\partial H}{\partial t}(t, \phi(t)) \Big/ \frac{\partial H}{\partial x}(t, \phi(t)).\end{aligned} \tag{4.4}$$

By (4.3), we have $H(t_0, \phi(t_0)) = H(t_0, x_0) = 0$, and by (4.4), we have

$$\frac{d}{dt}H(t, \phi(t)) = \frac{\partial H}{\partial t}(t, \phi(t)) + \phi'(t)\frac{\partial H}{\partial x}(t, \phi(t)) = 0.$$

Thus we have $H(t, \phi(t)) = 0$ on the interval $(t_0 - h, t_0 + h)$.

In case $N > 1$, we choose $a, r > 0$ so that $(t_0 - a, t_0 + a) \times \mathbb{B}(x_0, r) \subseteq U$ and so that the $N \times N$ matrix

$$D_x H = \left(\frac{\partial H_i}{\partial x_j}\right)_{i,j=1,2,\ldots,N}$$

is nonsingular on $(t_0 - a, t_0 + a) \times \mathbb{B}(x_0, r)$. Next, we replace (4.2) by

$$F(t, x) = -\Big[D_x H(t + t_0, x + x_0)\Big]^{-1}\left(\frac{\partial H}{\partial t}(t + t_0, x + x_0)\right).$$

The proof then proceeds as before. □

Remark 4.1.4 The proof of Theorem 4.1.3 given above is clearly limited to the case of one independent variable in the implicitly defined function. The case of one dependent variable and several independent variables can be obtained by replacing (4.2) with the appropriate system of first-order partial differential equations. The system of partial differential equations is solved by applying the existence theorem for ordinary differential equations (Theorem 4.1.1), *with parameters*, to each independent variable in turn. For example, if we have the equation $H(x, y, z) = 0$ which we are considering near a point (x_0, y_0, z_0) where H is zero and $\partial H/\partial z$ is nonzero (so the implicit function $z(x, y)$ will involve two independent variables), then there will be two first-order partial differential equations that $z(x, y)$ must satisfy:

$$\frac{\partial z}{\partial x} = -\frac{\partial H}{\partial x}\Big/\frac{\partial H}{\partial z}, \tag{4.5}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial H}{\partial y}\Big/\frac{\partial H}{\partial z}. \tag{4.6}$$

Restricting to a neighborhood of (x_0, y_0, z_0) in which $\partial H/\partial z$ is non-vanishing will enable us to conclude by an appeal to Rolle's theorem that the function $z(x, y)$ is uniquely defined, without the hypotheses for the uniqueness of solutions of ordinary differential equations.

The *second* equation, (4.6), is solved by solving the *first* equation, (4.5), with $y = y_0$ fixed and with the initial condition $z = z_0$. Then the resulting function $z(x, y_0)$ is used to provide the initial condition for (4.6); the initial value problem is then solved while treating x as a parameter. This process will produce a solution of (4.6) in an open set about the point (x_0, y_0). By carrying out the same process, but in the other order, we can obtain a solution of (4.5) in an open set about the same point (x_0, y_0). Because of the form of the right-hand sides of (4.5) and (4.6), those two solutions will be consistent and will define just one function that satisfies *both* equations.

Finally, the general implicit function theorem for any number of dependent and independent variables can be proved using Dini's induction procedure (see Section 3.2). □

We have seen that the implicit function theorem can be treated, in a sense, as a corollary of the existence theorem for ordinary differential equations. What we would like to do next is prove the converse: that we can use the implicit function theorem to prove the existence of solutions to ordinary differential equations. The Banach space methods of Section 3.4 will be required for this argument. We recall the statement of the theorem:

Theorem 4.1.1 *If $F(t, x)$, $(t, x) \in \mathbf{R} \times \mathbf{R}^N$, is continuous in the $(N + 1)$-dimensional region $(t_0 - a, t_0 + a) \times \mathbf{B}(x_0, r)$, then there exists a solution $x(t)$ of*

$$\frac{dx}{dt} = F(t, x), \quad x(t_0) = x_0, \tag{4.1}$$

defined over an interval $(t_0 - h, t_0 + h)$.

Proof. For convenience of notation, let us suppose that $t_0 = 0$.

Let $\mathcal{B}_0$ be the space of bounded continuous $\mathbf{R}^N$-valued functions on $(-a, a)$, normed by the supremum of the magnitude of the function. Let $\mathcal{B}_1$ be the space of bounded continuously differentiable $\mathbf{R}^N$-valued functions on $(-a, a)$ that also have a bounded derivative. We norm this space by the sum of the supremum of the magnitude of the function and the supremum of the magnitude of the derivative of the function. We define a map $\mathcal{F} : \mathcal{B}_1 \to \mathcal{B}_0 \times \mathbf{R}$ by setting

$$\mathcal{F}\Big[x(t)\Big] = \Big[x'(t) - F(t, x(t)),\ x(0) - x_0\Big].$$

With this notation, a solution of (4.1) is given by a zero of $\mathcal{F}$.

We imbed the problem of solving $\mathcal{F}[x] = [0, 0]$ into a larger problem Define $\mathcal{H} : [0, 1] \times \mathcal{B}_1 \to \mathcal{B}_0 \times \mathbf{R}$ by setting

$$\mathcal{H}\Big[\alpha,\ X(\tau)\Big] = \Big[X'(\tau) - \alpha F(\alpha\tau, X(\tau)),\ X(0) - x_0\Big].$$

We observe that

$$\mathcal{H}[0,\ x_0] = [0,\ 0], \tag{4.7}$$

where x_0 in (4.7) represents the constant function. Also, we observe that the Fréchet derivative of $\mathcal{H}$ at $[0, x_0]$ is given by $X \mapsto X'$. It follows from the implicit function theorem for Banach spaces, Theorem 3.4.10, that for all small enough choices of α there exists an $X(\alpha, \tau)$ such that

$$D_\tau X(\alpha, \tau) - \alpha F(\alpha\tau, X(\alpha, \tau)) = 0, \qquad X(\alpha, 0) = x_0.$$

For such an $\alpha > 0$, we define $x(t)$ by setting

$$x(t) = X(\alpha, t/\alpha).$$

It follows that

$$x'(t) = \frac{1}{\alpha} \quad D_\tau(\alpha, t/\alpha) = \frac{1}{\alpha} \cdot \alpha \cdot F[\alpha(t/\alpha), X(\alpha, t/\alpha)] = F(t, x(t)).$$

Thus our differential equation is solved, and the theorem is proved. □

4.2 Numerical Homotopy Methods

Suppose we wish to solve a system of nonlinear equations

$$F(x) = 0 \tag{4.8}$$

where $F : \mathbf{R}^N \to \mathbf{R}^N$ is smooth. Only in very special circumstances will it be possible to solve (4.8) in closed form; generally, numerical methods must be employed and an approximate solution thereby obtained. Of course, we would probably like to apply Newton's method, but for that we need a reasonable starting point for the iteration. In case we do not have such a reasonable starting point for Newton's method, some alternative procedure is needed. One such method is the *homotopy method* (also called the *continuation* or *imbedding* method).

In the homotopy method, we imbed the problem of interest, (4.8), into a larger problem of finding the zeros of a function $H : \mathbf{R}^{N+1} \to \mathbf{R}^N$ However, the function H is to be specially chosen so that the function $F_0 : \mathbf{R}^N \to \mathbf{R}^N$ defined by setting

$$F_0(x) = H(0, x) \tag{4.9}$$

is one that we understand well, while the function F in which we are interested is given by

$$F(x) = H(1, x).$$

The plan then is to follow the zeros of H from a starting point $(0, x_0) \in \mathbf{R}^{N+1}$ with $F_0(x_0) = 0$ along a curve $(t(s), x(s))$, $0 \le s \le 1$, for which

$$H(t(s), x(s)) = 0, \tag{4.10}$$